M-adaptation for the Acoustic Wave Equation

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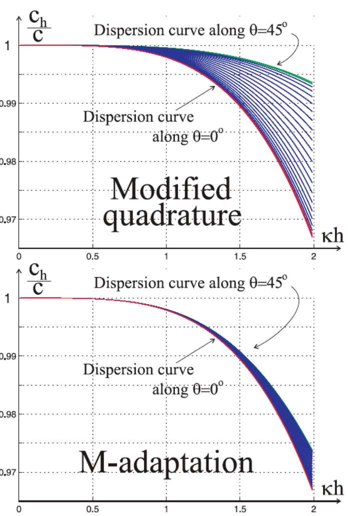
We have developed a novel adaptive strategy, dubbed m-adaptation, for solving the acoustic wave equation (in the time domain) on square meshes. The m-adaptation is based on selecting the optimal member of a three-parameter Mimetic Finite Difference (MFD) family of second-order schemes. This family contains as particular members the classical methods such as the finite element, finite difference, and a few other more recent methods. The optimal member of the MFD family eliminates the numerical dispersion at the fourth order and the numerical anisotropy at the sixth order. The numerical experiments show that the new approach is consistently better than the classical methods for reducing a long-time integration error.

Fig. 1. A more narrow band of values in the dispersion curves for the m-adaptation method (bottom) compared to the modified quadrature method [2] (top) for various angles θ between the planar wave and the mesh axis for the Courant number $\frac{c\Delta t}{h} = 0.75$ indicates smaller anisotropy.



Numerical modeling of wave propagation is essential for a large number of applied problems in acoustics, elasticity, and electromagnetics. The acoustic equation is one of the simplest examples of equation modeling wave propagation. For long integration times, the dominant contributions to an error in the solution come from such numerical artifacts as numerical dispersion and numerical anisotropy. The numerical dispersion is the phenomenon in which the propagation velocity of the wave in the numerical scheme depends on its wavelength, while in the continuum problem there is no such dependence. Typically, the effect of the numerical dispersion is greater on under-resolved waves with ten or fewer points per wavelength, making them travel slower than in the physical problem. As a consequence, the wave does not just arrive at a wrong time (which could be compensated by time rescaling) but it also has a highly distorted profile. The numerical anisotropy is the dependence of the numerical velocity of the wave on its orientation with respect to the mesh. For a 2D acoustic wave equation we developed an adaptation technique, dubbed m-adaptation, that selects an optimal member of a rich parameterized family of second-order methods with smallest (fourth-order) dispersion and (sixth-order) anisotropy.

The semi-discrete form of the acoustic wave equation in the time domain formulation is:

$$Mu_{tt} = Au \qquad (1)$$

where the mass and stiffness matrices M and A are assembled from elemental matrices M_E and A_E.

Since the mass matrix M has to be inverted on every time step, the explicit time discretization of equation (1) is computationally efficient only when the inverse M^{-1} is easy to compute. One of the common approaches is to replace the mass matrix M with a diagonal matrix D by lumping non-diagonal entries to the diagonal. This does not change the order of the numerical scheme but may lead to an undesirable increase of numerical dispersion. Another approach [1] is to replace the inverse M^{-1} with the product $D^{-1}MD^{-1}$, where the inverse is taken only for the diagonal matrix D. Similar to lumping, this approach does not change the order of the numerical scheme but may also result in the increase of the numerical dispersion. To compensate for the possible increase of the dispersion one can modify the stiffness and the mass matrices A and M using modified quadrature rules as is done in [1].

In the m-adaptation approach, we consider a parameterized Mimetic Finite Difference (MFD) family of numerical schemes from which we select a member with the smallest numerical dispersion and anisotropy [2]. The parameters in the MFD family appear through the elemental mass and stiffness matrices M_E^{MFD} and A_E^{MFD}, respectively. The elemental mass matrix M_E^{MFD} on a square element E depends on two parameters m_1, m_2 while the elemental stiffness matrix A_E depends on one parameter ζ.

The MFD family parameterized by (m_1, m_2, ζ) contains a large number of known methods as special cases–for example, standard Finite Difference (FD), rotated FD, weighted combination of standard and rotated FD, Finite Element (FE) with lumped mass matrix, and the modified quadrature method of Guddati and Yue [1]. Moreover, compared with the last method, the MFD family is richer–it contains one extra parameter.

For the acoustic wave equation in 2D the optimal parameters (m_1, m_2, ζ) can be selected based on the von-Neumann analysis. One

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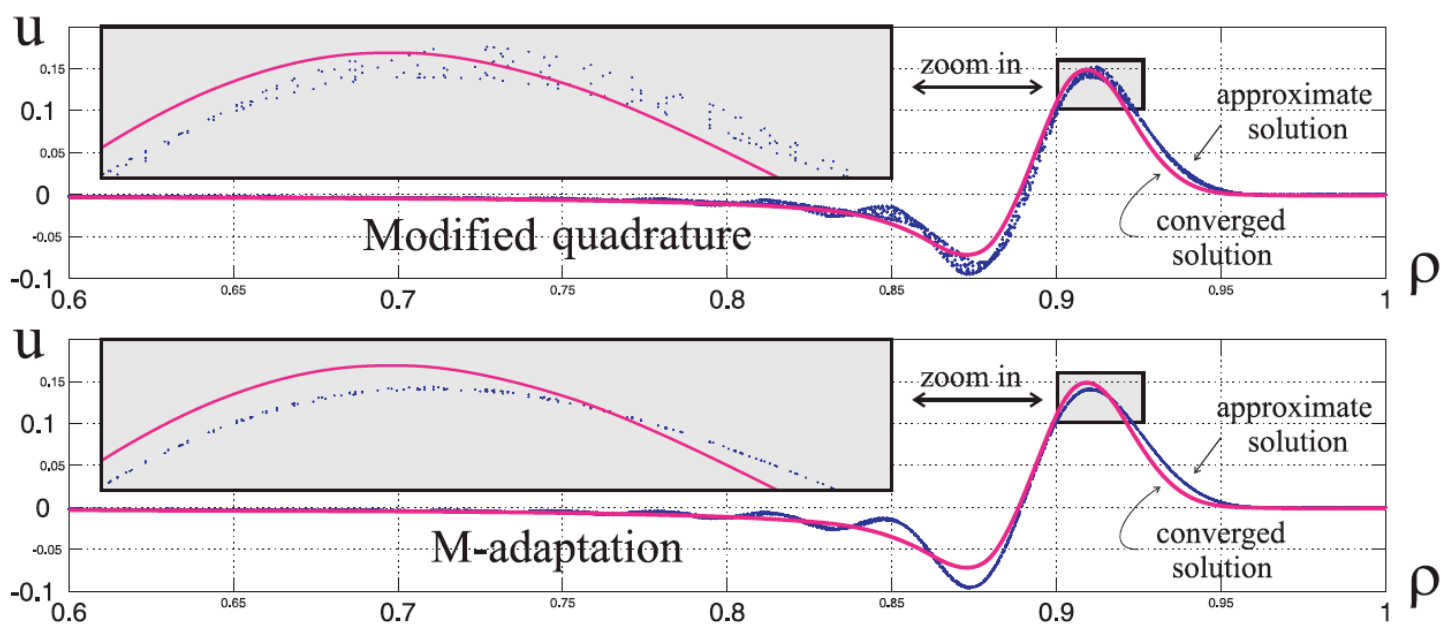

Fig. 2. Displacement as a function of the distance from the origin at time $T = 0.9$ obtained using the modified quadrature method (top) and the m-adaptation method (bottom) for a Gaussian initial displacement data.

obtains a local dispersion equation relating the numerical velocity of the wave c_h with its wave number κ, mesh size h, and the parameters (m_1, m_2, ζ). Expending the error between the physical and numerical velocities of the wave, $c - c_h$, in powers of wave resolution number, κh, we select the parameters (m_1, m_2, ζ) to eliminate the error at the leading powers of κh. As a result of m-adaptation, the numerical velocity c_h is accurate to the fourth order in dispersion (as in [2]) and to the sixth order in anisotropy (versus fourth order in[2] see Fig. 1 and 2).

In the future we plan to develop the m-adaptation technique for higher order schemes on general meshes and for elastic wave equations. The potential of m-adaptation is high because with increased order of the scheme and/or number of vertices in the element, the number of free parameters grows quadratically. This may lead to a dramatic improvement in the dispersion and anisotropy of the optimal scheme.

[1] Guddati, M.N. and B. Yue, *Comput Meth Appl Mech Eng* **193**, 275287 (2004).
[2] Gyrya, V. and K. Lipnikov, "M-adaptation Method for Solving Acoustic Wave Equation on Rectangular Meshes," *J Acoust Soc Am*, submitted (2012).

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